Exam Calculus of Variations and Optimal Control 2018-19

Date : 22-01-2019
Place : ACLO Station
Time : 09.00-12.00

The exam is OPEN BOOK; you can use all your books/papers/notes; but NO internet connection.
You are supposed to provide arguments to all your answers, and to explicitly refer to theorems/propositions whenever you use them.

1. Consider the minimization of

$$\int_0^T \left(x^T(t)Qx(t) + \dot{x}^T(t)R\dot{x}(t)\right) dt$$

over all functions $x : [0,T] \to \mathbb{R}^n$ satisfying $x(0) = x_0, x(T) = x_T$, where $Q = Q^T, R = R^T > 0$ are $n \times n$ matrices.

   (a) Write down the Euler-Lagrange equations.

   (b) Show that the unique solution to these Euler-Lagrange equations is a global minimum.

   (c) Write down the Beltrami identity for this case. How could this be also inferred from the Minimum principle?

   (d) Answer the questions of parts (a) and (b) for minimization over all $x : [0,T] \to \mathbb{R}^n$ only satisfying $x(0) = x_0$.

2. Solve the minimization of

$$\int_0^1 \ddot{x}^2(t) + 2tx(t)dt$$

over all functions $x : [0,1] \to \mathbb{R}$ satisfying $x(0) = 0, x(1) = 1$.

3. We want to move a mass in 2 seconds, beginning and ending with zero speed, using bounded acceleration. With x_1 its position and x_2 its speed, a model for this problem is

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \qquad x_1(0) = 0 \\ \dot{x}_2(t) &= u(t), \qquad x_2(0) = 0,\ x_2(2) = 0, \end{aligned}$$

with the acceleration $u(t)$ satisfying $u(t) \in [-1,1]$ for all t. We aim to maximize the traveled distance $x_1(2)$ at final time 2.

   (a) Determine the Hamiltonian $H(x,p,u)$.

   (b) Determine the Hamiltonian equations in $x(t)$ and $p(t)$ as used in Pontryagin's Minimum Principle, including all initial and final conditions.

   (c) Determine the general solution of the costate $p(t)$ for $t \in [0,2]$.

(d) Determine the optimal input $u(t)$ for $t \in [0, 2]$, and compute the maximal traveled distance $x_1(2)$.

4. The following model describes the interaction between a population of predators (with size x_1) and preys (with size x_2), and is (after scaling) given by the equations

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_1(t)x_2(t), \qquad x_1(0) \geq 0 \\ \dot{x}_2(t) &= x_2(t) - x_1(t)x_2(t), \qquad x_2(0) \geq 0 \end{aligned}$$

The first term on the right-hand side of the first equation shows that the predators will become extinct without food, while the second term shows that the growth of their population is proportional to the size of the population of prey. Likewise, the term on the right-hand side of the second equation shows that without predators, the population of prey will increase, and that its decrease is proportional to the size of the population of predators.

(a) Show that, apart from $(0, 0)$, the system has a second equilibrium point.

(b) Investigate the stability of both equilibrium points using linearization.

(c) Prove that the nonzero equilibrium point is stable using the function (with $\ln(x)$ denoting the natural logarithm)

$$V(x_1, x_2) = x_1 + x_2 - \ln(x_1x_2) - 2$$

Is it asymptotically stable?

5. Consider the scalar system

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

Determine the input $u_* : [0, \infty) \to \mathbb{R}$ which is such that $\lim_{t\to\infty} x_*(t) = 0$ and which is minimizing

$$\int_0^\infty u^4(t)dt$$

among all input functions $u : [0, \infty) \to \mathbb{R}$ for which their solutions $x(t)$ also satisfy $\lim_{t\to\infty} x(t) = 0$.

Distribution of points: Total 100; Free 10.

1. a: 5, b: 5, c: 5, d:. 5
2. 10
3. a: 5, b: 10, c: 5, d: 5
4. a: 5, b: 5, c: 10
5. 15